One-Loop Effective Action for Euclidean Maxwell Theory on Manifolds with Boundary

Giampiero Esposito,[1,2] [*] Alexander Yu. Kamenshchik[3] [†] and Klaus Kirsten[4] [‡]

[1] *Istituto Nazionale di Fisica Nucleare, Sezione di Napoli,*

Mostra d'Oltremare Padiglione 20, 80125 Napoli, Italy

[2] *Dipartimento di Scienze Fisiche,*

Mostra d'Oltremare Padiglione 19, 80125 Napoli, Italy

[3] *Nuclear Safety Institute, Russian Academy of Sciences,*

52 Bolshaya Tulskaya, Moscow 113191, Russia

[4] *Universität Leipzig, Institut für Theoretische Physik,*

Augustusplatz 10, 04109 Leipzig, Germany

Abstract

This paper studies the one-loop effective action for Euclidean Maxwell theory about flat four-space bounded by one three-sphere, or two concentric three-spheres. The analysis relies on Faddeev-Popov formalism and ζ-function regularization, and the Lorentz gauge-averaging term is used with magnetic boundary conditions. The contributions of transverse, longitudinal and normal modes of the electromagnetic potential, jointly with ghost modes, are derived in detail. The most difficult part of the analysis consists in the eigenvalue condition given by the determinant of a 2×2 or 4×4 matrix for longitudinal and normal modes. It is shown that the former splits into a sum

[*] Electronic address: esposito@napoli.infn.it

[†] Electronic address: grg@ibrae.msk.su

[‡] Electronic address: kirsten@tph100.physik.uni-leipzig.de

of Dirichlet and Robin contributions, plus a simpler term. This is the quantum cosmological case. In the latter case, however, when magnetic boundary conditions are imposed on two bounding three-spheres, the determinant is more involved. Nevertheless, it is evaluated explicitly as well. The whole analysis provides the building block for studying the one-loop effective action in covariant gauges, on manifolds with boundary. The final result differs from the value obtained when only transverse modes are quantized, or when noncovariant gauges are used.

I. INTRODUCTION

Over the last few years, a considerable effort has been produced in the literature to study the problem of boundary conditions in Euclidean quantum gravity and quantum cosmology [1–10]. This is motivated by the need to obtain a well defined path-integral representation of the $\langle \text{out} \mid \text{in} \rangle$ amplitudes and of the quantum state of the universe, and lies at the very heart of any attempt to understand the basic features of a theory of the quantized gravitational field. In particular, within the framework of the semiclassical approximation for the wave function of the universe, this analysis has led to the first calculation of one-loop divergences for massless spin-$\frac{1}{2}$ fields [11–16], Euclidean Maxwell theory [17–20], supergravity models [21] and Euclidean quantum gravity [6–10] in the presence of boundaries. Focusing on massless models, flat Euclidean backgrounds bounded by a three-sphere have been studied in detail for fields of various spins, with local or nonlocal boundary conditions, and in covariant as well as noncovariant gauges for gauge theories and gravitation [6–21]. For these fields the boundary conditions are mixed in that some components of the field obey one set of boundary conditions, and the remaining components obey another set of boundary conditions. This is indeed necessary to ensure invariance of the whole set of boundary conditions under infinitesimal gauge transformations, as well as their BRST invariance. For example, one may consider Euclidean Maxwell theory (which is the object of our investigation). In the classical theory, one may begin by fixing at the boundary the tangential components A_k of the electromagnetic potential. Such a boundary condition is invariant under infinitesimal gauge transformations ${}^{\xi}A_k = A_k + \partial_k \xi$ if and only if ξ itself vanishes at the boundary. In the semiclassical approximation of quantum theory, one expands about a vanishing background value for A_k, so that the electromagnetic potential reduces to pure perturbations $\mathcal{A}_k$, say. Moreover, if one follows the Faddeev-Popov method, one adds a gauge-averaging term $\frac{1}{2\alpha}[\Phi(\mathcal{A})]^2$ to the original Lagrangian, jointly with a ghost term which is necessary to ensure gauge invariance of the quantum theory. The first set of boundary conditions are now

$$\Big[\mathcal{A}_k\Big]_{\partial M} = 0, \tag{1.1}$$

$$[\varphi]_{\partial M} = 0, \tag{1.2}$$

where φ is a complex-valued ghost zero-form, corresponding to two *independent, real* ghost fields [22], which are both subject to homogeneous Dirichlet conditions at the boundary. At this stage, the only choice of boundary conditions on the normal component $\mathcal{A}_0$, whose gauge invariance is again guaranteed by the imposition of (1.2) on the ghost zero-form, is

$$\Big[\Phi(\mathcal{A})\Big]_{\partial M} = 0. \tag{1.3}$$

If the Lorentz gauge-averaging functional is chosen, Eqs. (1.1) and (1.3) lead to Robin conditions on $\mathcal{A}_0$, i.e.,

$$\left[\frac{\partial \mathcal{A}_0}{\partial \tau} + \mathcal{A}_0 \mathrm{Tr} K\right]_{\partial M} = 0, \tag{1.4}$$

where K is the extrinsic-curvature tensor of the boundary.

The analysis of conformal anomalies and one-loop divergences, however, is part of a more general program devoted to the investigation of the one-loop effective action on manifolds with boundary. As shown in Refs. [23–26], complete results are by now available for scalar and spin-$\frac{1}{2}$ fields. For Euclidean Maxwell theory, the contribution of transverse modes was obtained in Refs. [25,26], and the contribution of all perturbative modes in a noncovariant gauge was first obtained in Ref. [27]. For supergravity and quantum gravity, the contribution of transverse-traceless modes only has been obtained in Refs. [25,26]. It has been therefore our aim to present a detailed calculation of one-loop effective action in a covariant gauge including all perturbative modes of the problem in the presence of boundaries. This is necessary both for the sake of completeness, and to check whether the contributions of longitudinal, normal and ghost modes cancel each other exactly on such bounded regions. No such a cancellation was indeed found to occur in the analysis of conformal anomalies in Refs. [18–20].

Since the building block of our investigation is the study of real, massless scalar fields on the four-ball, subject to Dirichlet conditions, we find it helpful to present here a very brief outline of such a calculation. As shown in Refs. [23,28], the starting point is the integral

representation of the ζ-function of a self-adjoint, positive-definite elliptic operator by means of the Cauchy formula, which makes it possible to express $\zeta(s)$ of the Laplace operator in four dimensions as the sum

$$\zeta(s)=\sum_{l=1}^{\infty} l^2 Z_l(s)+\sum_{i=-1}^{3} A_i(s). \tag{1.5}$$

By using the notation in the Appendix for the uniform asymptotic expansions of Bessel functions, one has in (1.5)

$$Z_l(s)=\frac{\sin(\pi s)}{\pi}\int_0^{\infty} dz (zl/a)^{-2s}\frac{\partial}{\partial z}\Big[\ln I_l(lz)-l\eta+\ln\Big(\sqrt{2\pi l}(1+z^2)^{\frac{1}{4}}\Big)-\sum_{k=1}^{3}\frac{D_k(t)}{l^k}\Big], \tag{1.6}$$

$$A_{-1}(s)=\frac{1}{4\sqrt{\pi}}\frac{a^{2s}}{\Gamma(s)}\frac{\Gamma(s-\frac{1}{2})}{s}\zeta_R(2s-3), \tag{1.7}$$

$$A_0(s)=-\frac{1}{4}a^{2s}\zeta_R(2s-2), \tag{1.8}$$

$$A_1(s)=-\frac{a^{2s}}{\Gamma(s)}\zeta_R(2s-1)\sum_{j=0}^{1} x_{1,j}\frac{\Gamma(s+j+\frac{1}{2})}{\Gamma(j+\frac{1}{2})}, \tag{1.9}$$

$$A_2(s)=-\frac{a^{2s}}{\Gamma(s)}\zeta_R(2s)\sum_{j=0}^{2} x_{2,j}\frac{\Gamma(s+j+1)}{\Gamma(j+1)}, \tag{1.10}$$

$$A_3(s)=-\frac{a^{2s}}{\Gamma(s)}\zeta_R(2s+1)\sum_{j=0}^{3} x_{3,j}\frac{\Gamma(s+j+\frac{3}{2})}{\Gamma(j+\frac{3}{2})}, \tag{1.11}$$

where a is the radius of the three-sphere boundary. Thus, the algorithms described in Refs. [23,27] lead to the following result for the one-loop effective action:

$$\begin{aligned}\Gamma^{(1)} &= -\frac{1}{2}\zeta'(0)-\frac{1}{2}\zeta(0)\ln(\mu^2)\\ &= \frac{1}{360}\ln(\mu^2a^2)-\frac{1}{180}\ln(2)-\frac{173}{60480}-\frac{1}{6}\zeta_R'(-3)+\frac{1}{4}\zeta_R'(-2)-\frac{1}{12}\zeta_R'(-1).\end{aligned} \tag{1.12}$$

Section II evaluates the one-loop effective action for Euclidean Maxwell theory in a background motivated by quantum cosmology, i.e., flat Euclidean four-space bounded by a three-sphere. This results from the analysis of the wave function of the universe in the limit of

small three-geometries [29]. Magnetic boundary conditions in the Lorentz gauge, i.e., (1.1), (1.2) and (1.4), are imposed. Section III studies instead the occurrence of two concentric three-sphere boundaries. This case is more relevant for quantum field theory. Section IV presents some independent derivations, based on the technique developed by Barvinsky, Kamenshchik, Karmazin and Mishakov. Results and open problems are discussed in Sec. V, and relevant details are given in the Appendix.

II. ONE-BOUNDARY PROBLEM

In this section we study a background four-geometry given by flat Euclidean four-space bounded by a three-sphere. Since the boundary three-geometry is S^3, this ensures that the tangential components of the electromagnetic perturbations consist of a transverse part, $\mathcal{A}_k^T$, and a longitudinal part, $\mathcal{A}_k^L$, only [30]. These are expanded on a family of three-spheres centered on the origin as [31]

$$\mathcal{A}_k^T(x,\tau) = \sum_{n=2}^{\infty} f_n(\tau) S_k^{(n)}(x), \tag{2.1}$$

$$\mathcal{A}_k^L(x,\tau) = \sum_{n=2}^{\infty} g_n(\tau) P_k^{(n)}(x), \tag{2.2}$$

where τ is a radial coordinate $\in [0,a]$, x are local coordinates on S^3, and $S_k^{(n)}$ and $P_k^{(n)}$ are transverse and longitudinal vector harmonics on the three-sphere, respectively [32]. Moreover, the occurrence of the boundary with normal vector n^μ makes it possible to define the normal component of $A_\mu(x,\tau)$ as $A_0(x,\tau) \equiv n^\mu A_\mu(x,\tau)$. Its expansion on the same family of three-spheres reads [31]

$$\mathcal{A}_0(x,\tau) = \sum_{n=1}^{\infty} R_n(\tau) Q^{(n)}(x), \tag{2.3}$$

where $Q^{(n)}$ are scalar harmonics on S^3 [32]. We are actually facing a crucial point in our analysis, since the singularity at the origin of our background four-manifold calls into question the validity of the 3+1 split (2.1)–(2.3) *inside* the bounding three-sphere of radius

a [18–20]. In the analytic approach, one requires that the expansions (2.1)–(2.3) should match the boundary values $\mathcal{A}_k^T(x,a), \mathcal{A}_k^L(x,a), \mathcal{A}_0(x,a)$, and be regular $\forall \tau \in [0,a]$. This should pick out a unique smooth solution [17]. In the geometric analysis it seems that, as long as the operator acting on $\mathcal{A}^\mu$ reduces to $-g_{\mu\nu}\Box$ in flat four-space, the analysis of the problem remains well defined at any stage, since it does not contain any (explicit) reference to ill-defined objects or noncovariant elements (e.g., TrK terms in the differential operators). Indeed, in covariant gauges, the operator on $\mathcal{A}^\mu$ is

$$-g_{\mu\nu}\Box + R_{\mu\nu} + \Big(1 - \frac{1}{\alpha}\Big)\nabla_\mu \nabla_\nu,$$

and this reduces to the desired $-g_{\mu\nu}\Box$ in flat four-space, provided that one makes the Feynman choice for the α parameter: $\alpha_F = 1$.

The contribution of transverse modes $f_n(\tau)$ in (2.1) is independent of any choice of gauge-averaging term in the Faddeev-Popov Euclidean action, since such modes are decoupled from longitudinal modes (g_n) and normal modes (R_n). Thus, relying on the $\zeta_T'(0)$ value obtained in Refs. [25,26], one finds

$$\Gamma_T^{(1)} = \frac{77}{360}\ln(\mu^2 a^2) + \frac{29}{90}\ln(2) + \frac{1}{2}\ln(\pi) + \frac{6127}{30240} - \frac{1}{3}\zeta_R'(-3) + \frac{1}{2}\zeta_R'(-2) + \frac{5}{6}\zeta_R'(-1). \quad (2.4)$$

It is also straightforward to obtain the contribution of ghost modes. Bearing in mind that they behave as fermionic modes (if the gauge field is bosonic, as in our case), and imposing the boundary condition (1.2), one has simply to multiply by -2 the scalar-field result (1.12). This leads to

$$\Gamma_{\text{ghost}}^{(1)} = -\frac{1}{180}\ln(\mu^2 a^2) + \frac{1}{90}\ln(2) + \frac{173}{30240} + \frac{1}{3}\zeta_R'(-3) - \frac{1}{2}\zeta_R'(-2) + \frac{1}{6}\zeta_R'(-1). \quad (2.5)$$

The only technical difficulties consist in the analysis of coupled longitudinal and normal modes. As shown in Ref. [19], the boundary conditions (1.1) and (1.4) lead to an eigenvalue condition for such modes given by the vanishing of the determinant of a 2×2 matrix. With the notation of Ref. [19], in the Lorentz gauge this equation reads ($\forall n \geq 2$)

$$I_{n+1}(Ma)\Big(2\frac{I_{n-1}(Ma)}{(Ma/2)} + I_{n-2}(Ma) + I_n(Ma)\Big)$$

$$+\frac{(n+1)}{(n-1)}I_{n-1}(Ma)\Big(2\frac{I_{n+1}(Ma)}{(Ma/2)}+I_n(Ma)+I_{n+2}(Ma)\Big)=0. \tag{2.6}$$

Thus, the first step is to re-express $I_{n-2}, I_{n-1}, I_{n+1}$ and I_{n+2} in terms of I_n and I'_n only. By virtue of Eqs. (A16)–(A19) of the Appendix, and setting $Ma = zn$, Eq. (2.6) is found to involve the following function of z, I_n and I'_n:

$$\mathcal{F}=\frac{1}{z}I_n(zn)\Big(I_n(zn)+znI'_n(zn)\Big), \tag{2.7}$$

where proportionality parameters have been omitted, since they do not affect the calculation of $\zeta'(0)$. The function (2.7) should be inserted into the integral representation of the ζ-function for longitudinal (L) and normal (N) modes, i.e.,

$$\zeta_{LN}(s)=\frac{\sin(\pi s)}{\pi}\sum_{n=2}^{\infty}n^2\int_0^{\infty}dz(zn/a)^{-2s}\frac{\partial}{\partial z}\ln\Big[z^{-2n+1}\mathcal{F}\Big], \tag{2.8}$$

where n^2 is the degeneracy of such modes. Remarkably, after re-expressing the infinite sum in (2.8) as an infinite sum from 1 to ∞, minus the contribution of $n = 1$, one finds, by virtue of (2.7), that the resulting contribution to $\zeta'(0)$ is the sum of three contributions: the $\zeta'(0)$ value for a real scalar field subject to Dirichlet conditions on S^3; the $\zeta'(0)$ value for a real scalar field subject to Robin conditions on S^3, with u parameter equal to 1; the $\zeta'(0)$ value resulting from $n = 1$. The first is given in Ref. [23] and is encoded in the one-loop effective action (1.12). The second is evaluated in Ref. [33] as

$$\zeta'_{\mathrm{Robin},u=1}(0)=-\frac{41}{864}-\frac{7}{45}\ln(2)-\frac{1}{2}\ln(\pi)-\frac{31}{90}\ln(a)+\frac{1}{3}\zeta'_R(-3)+\frac{1}{2}\zeta'_R(-2)-\frac{11}{6}\zeta'_R(-1). \tag{2.9}$$

The third is obtained from (see (2.7) and (2.8))

$$\widetilde{\zeta}(s)\equiv-\frac{\sin(\pi s)}{\pi}\int_0^{\infty}dz(z/a)^{-2s}\frac{\partial}{\partial z}\ln\Big[z^{-2}I_1(z)(I_1(z)+zI'_1(z))\Big]. \tag{2.10}$$

This yields (see the Appendix)

$$\widetilde{\zeta}'(0)=\ln(\pi a^2). \tag{2.11}$$

By virtue of (1.12), (2.9) and (2.11) one finds

$$\zeta'_{LN}(0) = -\frac{631}{15120} - \frac{13}{90}\ln(2) + \frac{1}{2}\ln(\pi) + \frac{37}{45}\ln(a^2) + \frac{2}{3}\zeta'_R(-3) - \frac{5}{3}\zeta'_R(-1). \tag{2.12}$$

Bearing in mind that [19] $\zeta_{LN}(0) = \frac{37}{45}$, which is indeed encoded in (2.12), one obtains

$$\Gamma^{(1)}_{LN} = -\frac{37}{90}\ln(\mu^2 a^2) + \frac{13}{180}\ln(2) - \frac{1}{4}\ln(\pi) + \frac{631}{30240} - \frac{1}{3}\zeta'_R(-3) + \frac{5}{6}\zeta'_R(-1). \tag{2.13}$$

Last, the contribution of the decoupled normal mode R_1 should be considered. In the Lorentz gauge, R_1 takes the form [19]

$$R_1(\tau) = \frac{1}{\tau} I_2(M\tau), \tag{2.14}$$

up to an unessential multiplicative constant, and hence it contributes (see the Appendix)

$$\Gamma^{(1)}_{R_1} = \frac{3}{8}\ln(\mu^2 a^2) - \frac{1}{4}\ln(2) + \frac{1}{4}\ln(\pi). \tag{2.15}$$

One can now combine Eqs. (2.4), (2.5), (2.13) and (2.15) to find the full one-loop effective action in the Lorentz gauge as

$$\Gamma^{(1)} = \frac{31}{180}\ln(\mu^2 a^2) + \frac{7}{45}\ln(2) + \frac{1}{2}\ln(\pi) + \frac{6931}{30240} - \frac{1}{3}\zeta'_R(-3) + \frac{11}{6}\zeta'_R(-1). \tag{2.16}$$

Interestingly, this differs both from the result (2.4), which only involves transverse modes, and from the result for $\Gamma^{(1)}$ obtained in Ref. [27] in the noncovariant gauge $\nabla^\mu \mathcal{A}_\mu - \frac{2}{3}\mathcal{A}_0 \mathrm{Tr}K$:

$$\Gamma^{(1)}_{NC} = \frac{77}{360}\ln(\mu^2 a^2) - \frac{8}{45}\ln(2) + \frac{1}{4}\ln(\pi) + \frac{1991}{6048} - \frac{1}{3}\zeta'_R(-3) + \frac{5}{6}\zeta'_R(-1). \tag{2.17}$$

III. TWO-BOUNDARY PROBLEM

Section II has studied a background with boundary which is more relevant for quantum cosmology (at least in the Hartle-Hawking program), where one boundary three-surface shrinks to zero [1,2]. The standard quantum field theoretical framework, however, deals with boundary data on *two* boundary three-surfaces, which are necessary to specify completely the path-integral representation of the propagation amplitude [34]. Hence we here focus

on the one-loop analysis of Euclidean Maxwell theory in the presence of two concentric three-sphere boundaries [19].

Since the singularity at the origin ($\tau = 0$) is avoided in this boundary-value problem, the basis functions for the various modes in (2.1)–(2.3) become linear combinations of both I_n and K_n (modified) Bessel functions. We begin with the most difficult part of the calculation, i.e., the determinant of the 4×4 matrix which yields implicitly the eigenvalues for longitudinal and normal modes. Such a matrix, given in Eq. (3.13) of Ref. [19], is obtained by imposing the boundary conditions (1.1) and (1.4) at the three-sphere boundaries of radii r_- and r_+ (hereafter $r_+ > r_-$). Again, one has to express Bessel functions of various orders in terms of I_n, I'_n, K_n, K'_n only, where $n \geq 2$. After a lengthy calculation, such a determinant is found to take the form

$$\begin{aligned}\mathcal{D} = &\frac{16n^2}{(n-1)^2 M^2 r_- r_+}\Big(I_n(Mr_-)K_n(Mr_+) - I_n(Mr_+)K_n(Mr_-)\Big)\\ &\times \Big\{M^2 r_+ r_- \Big[I'_n(Mr_+)K'_n(Mr_-) - I'_n(Mr_-)K'_n(Mr_+)\Big]\\ &+ Mr_-\Big[I_n(Mr_+)K'_n(Mr_-) - I'_n(Mr_-)K_n(Mr_+)\Big]\\ &+ Mr_+\Big[I'_n(Mr_+)K_n(Mr_-) - I_n(Mr_-)K'_n(Mr_+)\Big]\\ &+ \Big[I_n(Mr_+)K_n(Mr_-) - I_n(Mr_-)K_n(Mr_+)\Big]\Big\}. \qquad (3.1)\end{aligned}$$

Thus, one has first to multiply (3.1) by M^2 to get rid of fake roots [35]. By virtue of the uniform asymptotic expansions (A1)–(A4), only the effects of $K_n(Mr_-)$ and $I_n(Mr_+)$ survive at large M [19]. Thus, after setting $Mr_+ = zn$ (cf. Sec. II), which implies that $Mr_- = znr_-/r_+$, the contributions of the A_i functions to $\zeta'(0)$ (see (1.5)) can be obtained by studying

$$\begin{aligned}\ln \mathcal{D} \sim &\ln\Big[z^{-n} I_n(zn)\Big] + \ln\Big[z^n K_n(znr_-/r_+)\Big]\\ &+ \ln\Big[I_n(zn) + znI'_n(zn)\Big]\\ &+ \ln\Big[K_n(znr_-/r_+) + zn\frac{r_-}{r_+}K'_n(znr_-/r_+)\Big]. \qquad (3.2)\end{aligned}$$

This means that the asymptotic terms are a sum of Dirichlet contributions for the inner and outer space, and Robin contributions for the inner and outer space with $u = 1$. Looking

at the asymptotics of K_n and K_n', the relation between inner space and outer space is immediate: $(A_{-1})_I = -(A_{-1})_K$, $(A_0)_I = (A_0)_K$, $(A_i)_I = (-1)^i (A_i)_K$ (see Eqs. (A12)–(A15)). The resulting asymptotics of (3.2) reads

$$A_{-1}(s) = \frac{(r_+^{2s} - r_-^{2s})}{2\sqrt{\pi}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s+1)} \zeta_H(2s-3;2), \tag{3.3}$$

$$A_0(s) = 0, \tag{3.4}$$

$$A_i(s) = -\frac{1}{\Gamma(s)} \left(r_+^{2s} - r_-^{2s}\right) \zeta_H(2s+i-2;2) \sum_{l=0}^{i} (x_{i,l} + z_{i,l}) \frac{\Gamma(s+l+\frac{i}{2})}{\Gamma(l+\frac{i}{2})}, \ i = 1, 3, \tag{3.5}$$

$$A_2(s) = -\frac{1}{\Gamma(s)} \left(r_+^{2s} + r_-^{2s}\right) \zeta_H(2s;2) \sum_{l=0}^{2} (x_{2,l} + z_{2,l}) \frac{\Gamma(s+l+1)}{\Gamma(l+1)}. \tag{3.6}$$

Note that the second argument of the Hurwitz ζ-function is 2, to take into account that the infinite sums defining A_i start from $n = 2$ in our problem. Moreover, the $x_{i,l}$ and $z_{i,l}$ are the coefficients of the polynomials D_i (Dirichlet case) and M_i (Robin case), respectively (see Eqs. (A6)–(A11)). In Eqs. (3.3)–(3.6) one has now to pick out the coefficients of the terms linear in s, since these are the only ones which contribute to $\zeta'(0)$. Hence one finds

$$A'_{-1}(0) = \frac{119}{60} \ln(r_+/r_-), \tag{3.7}$$

$$A'_0(0) = A'_1(0) = 0, \tag{3.8}$$

$$A'_2(0) = -\frac{3}{2}, \tag{3.9}$$

$$A'_3(0) = -\frac{61}{180} \ln(r_+/r_-), \tag{3.10}$$

which imply

$$\sum_{i=-1}^{3} A'_i(0) = -\frac{3}{2} + \frac{74}{45} \ln(r_+/r_-). \tag{3.11}$$

The contribution of $Z(s) \equiv \sum_{n=2}^{\infty} n^2 Z_n(s)$ to $\zeta'(0)$ (cf. (1.6)) involves the logarithm of the Bessel terms in (3.1), and a further contribution resulting from the uniform asymptotics of such Bessel functions. Hence it reads (see the Appendix for details)

$$
\begin{aligned}
Z'(0) = -\sum_{n=2}^{\infty} n^2 \Big\{ & \ln\Big[-I_n(Mr_-)K_n(Mr_+) + I_n(Mr_+)K_n(Mr_-)\Big] \\
& + \ln\Big[I_n(Mr_+)K_n(Mr_-) - I_n(Mr_-)K_n(Mr_+) \\
& + Mr_- \Big[I_n(Mr_+)K_n'(Mr_-) - I_n'(Mr_-)K_n(Mr_+)\Big] \\
& + Mr_+ \Big[I_n'(Mr_+)K_n(Mr_-) - I_n(Mr_-)K_n'(Mr_+)\Big] \\
& + M^2 r_+ r_- \Big[I_n'(Mr_+)K_n'(Mr_-) - I_n'(Mr_-)K_n'(Mr_+)\Big]\Big] \\
& - 2n\Big[\eta(Mr_+) - \eta(Mr_-)\Big] + \frac{1}{n^2}\Big\},
\end{aligned}
\tag{3.12}
$$

where all Bessel functions should be studied in the limit as $M \to 0$ [23]. One can thus use Eqs. (A20) and (A21) which express the limiting behavior of Bessel functions in such a case. Many terms are then found to cancel each other exactly, leading to

$$
Z'(0) = -\sum_{n=2}^{\infty} n^2 \Big\{ 2\ln\Big[1 - (r_-/r_+)^{2n}\Big] + \ln\Big(1 - \frac{1}{n^2}\Big) + \frac{1}{n^2}\Big\},
\tag{3.13}
$$

where one has [27]

$$
-\sum_{n=2}^{\infty} n^2 \Big[\ln\Big(1 - \frac{1}{n^2}\Big) + \frac{1}{n^2}\Big] = \frac{3}{2} - \ln(\pi).
\tag{3.14}
$$

Equations (3.11), (3.13) and (3.14) imply that

$$
\zeta_{LN}'(0) = \frac{74}{45}\ln(r_+/r_-) - \ln(\pi) - 2\sum_{n=2}^{\infty} n^2 \ln\Big[1 - (r_-/r_+)^{2n}\Big].
\tag{3.15}
$$

Let us now study the decoupled normal mode R_1. In our two-boundary problem, it takes the form

$$
R_1(\tau) = \frac{\beta_1}{\tau} I_2(M\tau) + \frac{\beta_2}{\tau} K_2(M\tau),
\tag{3.16}
$$

where β_1 and β_2 are some constants. By virtue of the boundary condition (1.4), one should set to zero at the three-sphere boundaries the linear combination $\frac{dR_1}{d\tau} + \frac{3}{\tau}R_1$. The resulting eigenvalue condition has no fake roots, and hence one finds (see Sec. IV)

$$\Gamma_{R_1}^{(1)} = -\frac{1}{4}\ln(\mu^2 r_+ r_-) + \frac{1}{2}\ln(r_+/r_-) + \frac{1}{2}\ln\Big[1-(r_-/r_+)^2\Big]. \tag{3.17}$$

Last, we study the determinants of the 2×2 matrices which yield implicitly the eigenvalues for transverse and ghost modes. In both cases, the eigenvalue condition is

$$I_n(Mr_-)K_n(Mr_+) - I_n(Mr_+)K_n(Mr_-) = 0, \tag{3.18}$$

where for ghost modes the integer n is ≥ 1 and the degeneracy is $-2n^2$ [19], and for transverse modes the integer n is ≥ 2, with degeneracy $2(n^2-1)$ [31]. Bearing in mind the ghost degeneracy and Eqs. (1.7)–(1.11), the asymptotic contribution for ghosts is expressed by

$$A_{-1}^{\mathrm{gh}}(s) = -\frac{1}{2\sqrt{\pi}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s+1)}\Big[r_+^{2s} - r_-^{2s}\Big]\zeta_R(2s-3), \tag{3.19}$$

$$A_0^{\mathrm{gh}}(s) = \frac{1}{2}\Big[r_+^{2s} + r_-^{2s}\Big]\zeta_R(2s-2), \tag{3.20}$$

$$A_i^{\mathrm{gh}}(s) = \frac{2}{\Gamma(s)}\Big[r_+^{2s} + (-1)^i r_-^{2s}\Big]\zeta_R(2s+i-2)\sum_{l=0}^{i} x_{i,l}\frac{\Gamma(s+l+\frac{i}{2})}{\Gamma(l+\frac{i}{2})}, \tag{3.21}$$

while for transverse modes one finds

$$A_{-1}^{\mathrm{tr}}(s) = \frac{1}{2\sqrt{\pi}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s+1)}\Big[r_+^{2s} - r_-^{2s}\Big]\Big[\zeta_R(2s-3) - \zeta_R(2s-1)\Big], \tag{3.22}$$

$$A_0^{\mathrm{tr}}(s) = -\frac{1}{2}\Big[r_+^{2s} + r_-^{2s}\Big]\Big[\zeta_R(2s-2) - \zeta_R(2s)\Big], \tag{3.23}$$

$$A_i^{\mathrm{tr}}(s) = -\frac{2}{\Gamma(s)}\Big[r_+^{2s} + (-1)^i r_-^{2s}\Big]\Big[\zeta_R(2s+i-2) - \zeta_R(2s+i)\Big]\sum_{l=0}^{i} x_{i,l}\frac{\Gamma(s+l+\frac{i}{2})}{\Gamma(l+\frac{i}{2})}. \tag{3.24}$$

The most convenient way to proceed is now to add up the contributions (3.19)–(3.21) and (3.22)–(3.24). This yields

$$A_{-1}(s) = -\frac{1}{2\sqrt{\pi}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s+1)}\Big[r_+^{2s} - r_-^{2s}\Big]\zeta_R(2s-1), \tag{3.25}$$

$$A_0(s) = \frac{1}{2}\Big[r_+^{2s} + r_-^{2s}\Big]\zeta_R(2s), \tag{3.26}$$

$$A_i(s) = \frac{2}{\Gamma(s)}\Big[r_+^{2s} + (-1)^i r_-^{2s}\Big]\zeta_R(2s+i)\sum_{l=0}^{i} x_{i,l}\frac{\Gamma(s+l+\frac{i}{2})}{\Gamma(l+\frac{i}{2})}. \tag{3.27}$$

As in the previous cases, the contributions to $\zeta'(0)$ of (3.25)–(3.27) are obtained by considering their expansion in the neighboorhood of $s=0$, and adding the coefficients of all terms linear in s. This yields

$$\sum_{i=-1}^{3} A_i'(0) = -\frac{1}{3}\ln(r_+/r_-) - \frac{1}{2}\ln(4\pi^2 r_+ r_-). \tag{3.28}$$

Moreover, the form of $Z'(0)$ for ghost and transverse modes is considerably simplified because the full degeneracy is -2. This leads to (cf. Eqs. (3.12) and (3.13))

$$\begin{aligned} Z^{\mathrm{gh,tr}'}(0) &= 2\sum_{n=1}^{\infty}\Big\{\ln\Big[-I_n(Mr_-)K_n(Mr_+) + I_n(Mr_+)K_n(Mr_-)\Big] \\ &\quad - n\Big[\eta(Mr_+) - \eta(Mr_-)\Big] + \ln(-2n)\Big\} \\ &= 2\sum_{n=1}^{\infty}\ln\Big[1-(r_-/r_+)^{2n}\Big]. \end{aligned} \tag{3.29}$$

Last, since ghost modes yield a vanishing contribution to $\zeta(0)$, while transverse modes contribute $-\frac{1}{2}$, one finds

$$\Gamma^{(1)}_{\mathrm{gh,tr}} = \frac{1}{4}\ln(\mu^2 r_+ r_-) - \sum_{n=1}^{\infty}\ln\Big[1-(r_-/r_+)^{2n}\Big] + \frac{1}{6}\ln(r_+/r_-) + \frac{1}{2}\ln(2\pi). \tag{3.30}$$

The results (3.15), (3.17) and (3.30) lead to the following value of the one-loop effective action in the two-boundary problem:

$$\begin{aligned} \Gamma^{(1)} &= -\frac{7}{45}\ln(r_+/r_-) + \frac{1}{2}\ln(2) + \ln(\pi) - \frac{1}{2}\ln\Big[1-(r_-/r_+)^2\Big] \\ &\quad + \sum_{n=1}^{\infty}(n^2-1)\ln\Big[1-(r_-/r_+)^{2n}\Big]. \end{aligned} \tag{3.31}$$

IV. BARVINSKY-KAMENSHCHIK-KARMAZIN-MISHAKOV TECHNIQUE

When a series of difficult calculations is performed, it is appropriate to have an independent check of the final result. For this purpose, we here outline the application of the technique described in Refs. [14,15,35]. Let f_n be the function occurring in the equation

obeyed by the eigenvalues by virtue of boundary conditions, and let $d(n)$ be the degeneracy of such eigenvalues labelled by the integer n. One then defines the function

$$I(M^2, s) \equiv \sum_{n=n_0}^{\infty} d(n) n^{-2s} \ln f_n(M^2), \tag{4.1}$$

where M^2 is related to the eigenvalues through the relation $M^2 = -\lambda_n$, and fake roots (e.g., $M = 0$ for Bessel functions) have been taken out when f_n is written down. The function (4.1) admits an analytic continuation to the complex-s plane as a meromorphic function with a simple pole at $s = 0$: i.e.,

$$\text{“}I(M^2, s)\text{”} = \frac{I_{\text{pole}}(M^2)}{s} + I^R(M^2) + \mathrm{O}(s). \tag{4.2}$$

The functions occurring on the right-hand side of (4.2) make it possible to evaluate $\zeta(0)$ and $\zeta'(0)$ in quantum field theory as

$$\zeta(0) = I_{\log} + I_{\text{pole}}(M^2 = \infty) - I_{\text{pole}}(M^2 = 0), \tag{4.3}$$

$$\zeta'(0) = I^R(M^2 = \infty) - I^R(M^2 = 0) - \int_0^{\infty} \ln(M^2) \frac{dI_{\text{pole}}(M^2)}{dM^2} dM^2, \tag{4.4}$$

where $I_{\log}$ is the coefficient of $\ln(M)$ in the uniform asymptotic expansion of $I(M^2, s)$, after taking out fake roots. In quantum mechanics, as well as for decoupled modes of a quantum field, $\zeta(0)$ reduces to $I_{\log}$, and $\zeta'(0)$ reduces to $I^R(\infty) - I^R(0)$.

In Eq. (3.16), the decoupled mode gives rise to the eigenvalue condition

$$\begin{aligned} 0 = &\left(I_2'(Mr_+) + 2\frac{I_2(Mr_+)}{Mr_+} \right) \left(K_2'(Mr_-) + 2\frac{K_2(Mr_-)}{Mr_-} \right) \\ - &\left(I_2'(Mr_-) + 2\frac{I_2(Mr_-)}{Mr_-} \right) \left(K_2'(Mr_+) + 2\frac{K_2(Mr_+)}{Mr_+} \right). \end{aligned} \tag{4.5}$$

Thus, Eqs. (4.1), (4.2) and (A1)–(A4) lead to

$$I^R(\infty) = -\ln(2) - \frac{1}{2}\ln(r_+ r_-). \tag{4.6}$$

Moreover, $I^R(0)$ is obtained from the limiting behavior of Bessel functions as $M \to 0$ (see (A20) and (A21)), and one finds

$$I^R(0) = -\ln(2) + \ln(r_+/r_-) + \ln\Big(1 - (r_-/r_+)^2\Big). \tag{4.7}$$

The result (3.17) is obtained if one bears in mind that the decoupled mode provides an example of a nontrivial eigenfunction obeying the boundary conditions and belonging to the zero eigenvalue. Hence one deals with a one-dimensional null space whose dimension should be added to $I_{\log}$ to obtain the correct $\zeta(0)$ value for R_1 as $\zeta(0) = \frac{1}{2}$ [19].

It is also instructive to outline the ghost calculation in the two-boundary problem. The function of Eq. (4.1) takes then the form

$$I(M^2, s) = 2\sum_{n=1}^{\infty} n^{2-2s} \ln\Big[I_n(nMr_+)K_n(nMr_-) - I_n(nMr_-)K_n(nMr_+)\Big]. \tag{4.8}$$

As $M \to \infty$, only $K_n(nMr_-)$ and $I_n(nMr_+)$ contribute to $\zeta'(0)$, and the uniform asymptotic expansions (A1) and (A3) imply that

$$I^R(\infty) = 0, \tag{4.9}$$

since $\zeta_R(-2) = 0$. To evaluate $I^R(0)$ one needs instead the limiting behavior of Bessel functions as $M \to 0$. Thus, the expansions (A20) and (A21) lead to

$$I^R(0) = -2\sum_{n=1}^{\infty} n^2 \ln(n) + \frac{1}{60}\ln(r_+/r_-) + 2\sum_{n=1}^{\infty} n^2 \ln\Big[1 - (r_-/r_+)^{2n}\Big]. \tag{4.10}$$

Last, the third term on the right-hand side of (4.4) is found to be

$$-\int_0^{\infty} \ln(M^2)\frac{dI_{\text{pole}}}{dM^2}dM^2 = -\frac{1}{180}\ln(r_+/r_-). \tag{4.11}$$

Equations (4.9)–(4.11) yield

$$\zeta'(0)_{\text{gh}} = -\frac{1}{45}\ln(r_+/r_-) - 2\sum_{n=1}^{\infty} n^2 \ln\Big[1 - (r_-/r_+)^{2n}\Big] - 2\zeta_R'(-2), \tag{4.12}$$

and this should be multiplied by -1, since the ghost contribution has fermionic nature for a bosonic field.

For transverse modes, an analogous procedure yields

$$\begin{aligned}\zeta'(0)_{\text{tr}} = &-\frac{16}{45}\ln(r_+/r_-) - \frac{1}{2}\ln(r_+r_-) - 2\sum_{n=1}^{\infty}(n^2-1)\ln\Big[1 - (r_-/r_+)^{2n}\Big] \\ &- \ln(2\pi) - 2\zeta_R'(-2).\end{aligned} \tag{4.13}$$

Equations (4.12) and (4.13) yield a result in complete agreement with Eq. (3.30).

Last, to evaluate the contribution of longitudinal and normal modes one starts from the determinant (3.1), which implies

$$I^R(\infty) = \sum_{n=2}^{\infty} n^2 \ln \frac{n^2}{(n-1)^2 r_+ r_-}, \tag{4.14}$$

$$I^R(0) = \sum_{n=2}^{\infty} \frac{1}{r_+ r_-} \left[(r_+/r_-)^{2n} - (r_-/r_+)^{2n} \right]^2 \frac{(n^2-1)}{(n-1)^2}. \tag{4.15}$$

Moreover, the third term on the right-hand side of (4.4) contributes

$$-\int_0^{\infty} \ln(M^2) \frac{dI_{\rm pole}}{dM^2} dM^2 = -\frac{61}{180} \log(r_+/r_-). \tag{4.16}$$

The results (4.14)–(4.16), jointly with (3.14), lead to $\zeta'_{LN}(0)$ as in (3.15).

V. RESULTS AND OPEN PROBLEMS

Our paper has studied in detail the one-loop effective action $\Gamma^{(1)}$ for Euclidean Maxwell theory on manifolds with boundary, in the case of flat four-space bounded by a three-sphere, or two concentric three-spheres. Our main result is that, by using covariant gauges such as the Lorentz gauge within the framework of Faddeev-Popov formalism for semiclassical amplitudes, $\Gamma^{(1)}$ (see (2.16)) differs from the contribution of transverse modes obtained in Refs. [25,26] (see (2.4)), and it also differs from the value obtained in Ref. [27] in the noncovariant gauge $\nabla^\mu \mathcal{A}_\mu - \frac{2}{3}\mathcal{A}_0 {\rm Tr} K$ (see (2.17)). This is confirmed by the two-boundary analysis of Sec. III (see Eq. (3.31)). Our result seems to add evidence in favor of longitudinal, normal and ghost modes not being able to cancel each other's effects exactly on manifolds with boundary [17–20].

The extension to curved backgrounds (e.g., S^4 bounded by S^3) is of purely technical nature and can be obtained after dealing properly with the asymptotics of Legendre functions (instead of the Bessel functions occurring in the flat case). At least three outstanding problems, however, remain. First, one would like to evaluate the one-loop effective action

for the family of noncovariant gauges $\nabla^\mu \mathcal{A}_\mu - b\mathcal{A}_0 \mathrm{Tr}K$, first introduced in Ref. [17], and then studied extensively in Refs. [19,20]. These gauges have been criticized in Ref. [36], but they appear a necessary step to complete the quantization program in arbitrary gauges, at least in the semiclassical approximation.

Second, one would like to apply our algorithms to the analysis of Euclidean quantum gravity. For this purpose, one can impose the boundary conditions in terms of (complementary) projectors proposed in Ref. [5], or the boundary conditions completely invariant under infinitesimal diffeomorphisms [4], or the boundary conditions of Ref. [8], which are Robin on h_{ij} and the ghost one-form, and Dirichlet on normal components h_{00} and h_{0i} of metric perturbations. Yet another possibility is represented by the nonlocal boundary conditions of Ref. [9]. In such cases, it would be interesting to investigate the asymptotics of the eigenvalue condition given by the vanishing of the determinant of an 8×8 matrix in the two-boundary problem for pure gravity. More work can be done in this respect.

Last, but not least, geometric formulas for $\zeta'(0)$ are still lacking in arbitrary gauges on manifolds with boundary. What happens is that the usual Schwinger-DeWitt method fails to hold for nonminimal operators resulting from the choice of arbitrary gauge-averaging terms in the Euclidean action. More precisely, the factor which stands before the series in t in the heat-kernel asymptotics is not a Gaussian but a complicated special function. This leads in turn to relations for heat-kernel coefficients unbounded from below as well as above, and hence these equations cannot be solved recursively [37]. Nevertheless, if one were able to generalize the technique developed in Refs. [38,39] to manifolds with boundary, one would obtain an independent check of the several analytic results which can be derived in the near future. This would lead in turn to a much deeper understanding of heat-kernel asymptotics for quantized gauge fields and quantum gravity on manifolds with boundary.

Such an investigation is regarded by the authors as an important task for the years to come, and it makes us feel that a new exciting age is in sight in the application of heat-kernel methods to quantum field theory.

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APPENDIX:

The uniform asymptotic expansions as $\rho \to \infty$ of the Bessel functions $I_\rho(\rho z), K_\rho(\rho z)$, jointly with their first derivatives, are derived in detail in Ref. [40], and they play a fundamental role in the analytic investigation of conformal anomalies and one-loop effective action. In terms of the Debye polynomials $u_k(t)$ and $v_k(t)$ [41], they read

$$I_\rho(\rho z) \sim \frac{1}{\sqrt{2\pi\rho}} \frac{e^{\rho\eta}}{(1+z^2)^{\frac{1}{4}}} \left[1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\rho^k}\right], \tag{A1}$$

$$I'_\rho(\rho z) \sim \frac{1}{\sqrt{2\pi\rho}} e^{\rho\eta} \frac{(1+z^2)^{\frac{1}{4}}}{z} \left[1 + \sum_{k=1}^{\infty} \frac{v_k(t)}{\rho^k}\right], \tag{A2}$$

$$K_\rho(\rho z) \sim \sqrt{\frac{\pi}{2\rho}} \frac{e^{-\rho\eta}}{(1+z^2)^{\frac{1}{4}}} \left[1 + \sum_{k=1}^{\infty} (-1)^k \frac{u_k(t)}{\rho^k}\right], \tag{A3}$$

$$K'_\rho(\rho z) \sim -\sqrt{\frac{\pi}{2\rho}} e^{-\rho\eta} \frac{(1+z^2)^{\frac{1}{4}}}{z} \left[1 + \sum_{k=1}^{\infty} (-1)^k \frac{v_k(t)}{\rho^k}\right], \tag{A4}$$

where $t \equiv \frac{1}{\sqrt{1+z^2}}$, and $\eta \equiv \sqrt{1+z^2} + \ln[z/(1+\sqrt{1+z^2})]$. In the one-loop analysis it is necessary to evaluate the logarithm of the equation obeyed by the eigenvalues by virtue of boundary conditions. In particular, we need the asymptotic expansion [27]

$$\ln\left[1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\rho^k}\right] \sim \sum_{p=1}^{\infty} \frac{D_p(t)}{\rho^p}, \tag{A5}$$

where [27]

$$D_1(t) = \frac{1}{8}t - \frac{5}{24}t^3, \tag{A6}$$

$$D_2(t) = \frac{1}{16}t^2 - \frac{3}{8}t^4 + \frac{5}{16}t^6, \tag{A7}$$

$$D_3(t) = \frac{25}{384}t^3 - \frac{531}{640}t^5 + \frac{221}{128}t^7 - \frac{1105}{1152}t^9. \tag{A8}$$

In the case of Robin boundary conditions, a linear combination of I_ρ and I'_ρ is set to zero at the boundary, and a dimensionless parameter u occurs in the eigenvalue condition. Thus, the polynomials (A6)–(A8) are replaced by [27]

$$M_1(t,u) = \Big(-\frac{3}{8} + u\Big)t + \frac{7}{24}t^3, \tag{A9}$$

$$M_2(t,u) = \Big(-\frac{3}{16} + \frac{1}{2}u - \frac{1}{2}u^2\Big)t^2 + \Big(\frac{5}{8} - \frac{1}{2}u\Big)t^4 - \frac{7}{16}t^6, \tag{A10}$$

$$\begin{aligned} M_3(t,u) = {} & \Big(-\frac{21}{128} + \frac{3}{8}u - \frac{1}{2}u^2 + \frac{1}{3}u^3\Big)t^3 + \Big(\frac{869}{640} - \frac{5}{4}u + \frac{1}{2}u^2\Big)t^5 \\ & + \Big(-\frac{315}{128} + \frac{7}{8}u\Big)t^7 + \frac{1463}{1152}t^9. \end{aligned} \tag{A11}$$

When also K_ρ functions occur in the calculation of functional determinants, one has polynomials $\widetilde{D}_i(t) = (-1)^i D_i(t)$, and $\widetilde{M}_i(t,u) = (-1)^i M_i(t,u)$.

In the case of Dirichlet boundary conditions, the functions (1.7)–(1.11) are infinite sums of the contributions [23,27,28]

$$A_{-1}^l = \frac{\sin(\pi s)}{\pi} \int_0^\infty dz (zl/a)^{-2s} \frac{\partial}{\partial z} \ln\left(\frac{z^{-l}}{\sqrt{2\pi l}} e^{l\eta}\right), \tag{A12}$$

$$A_0^l = \frac{\sin(\pi s)}{\pi} \int_0^\infty dz (zl/a)^{-2s} \frac{\partial}{\partial z} \ln(1+z^2)^{-\frac{1}{4}}, \tag{A13}$$

$$A_i^l = \frac{\sin(\pi s)}{\pi} \int_0^\infty dz (zl/a)^{-2s} \frac{\partial}{\partial z} \left(\frac{D_i(t)}{l^i}\right). \tag{A14}$$

In the two-boundary problems, however, also K functions and their first derivatives contribute. By virtue of (A3), (A4), (A12)–(A14) one thus finds

$$(A_{-1})_I = -(A_{-1})_K \ , \ (A_0)_I = (A_0)_K \ , \ (A_i)_I = (-1)^i (A_i)_K, \tag{A15}$$

for both Dirichlet and Robin boundary conditions. This leads to (3.3)–(3.6).

The recurrence relations among Bessel functions used in the course of deriving (2.7) from (2.6) are

$$I_{n-2}(z) = \Big(1 + \frac{2n(n-1)}{z^2}\Big) I_n(z) + 2\frac{(n-1)}{z} I'_n(z), \tag{A16}$$

$$I_{n-1}(z) = I'_n(z) + \frac{n}{z} I_n(z), \tag{A17}$$

$$I_{n+1}(z) = I'_n(z) - \frac{n}{z} I_n(z), \tag{A18}$$

$$I_{n+2}(z) = \Big(1 + \frac{2n(n+1)}{z^2}\Big) I_n(z) - 2\frac{(n+1)}{z} I'_n(z). \tag{A19}$$

The limiting behavior of Bessel functions as $z \to 0$, which is necessary to deal properly with (3.12), is

$$I_n(z) \sim \frac{(z/2)^n}{\Gamma(n+1)}, \tag{A20}$$

$$K_n(z) \sim \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \frac{(n-k-1)!}{k!(z/2)^{n-2k}}. \tag{A21}$$

The contribution (2.15) to the one-loop effective action is the contribution of the decoupled normal mode R_1, and it is best tackled in terms of the algorithm of Ref. [35]. The general structure of $\zeta'(0)$ is then (see Sec. IV) $\zeta'(0) = I^R(\infty) - I^R(0)$, where

$$I^R(\infty) = -\frac{1}{2}\ln(a) - \frac{1}{2}\ln(2) - \frac{1}{2}\ln(\pi), \tag{A22}$$

$$I^R(0) = \ln(a) - \ln(2). \tag{A23}$$

Moreover, $I_{\log} = -\frac{3}{4}$ [19]. Hence one gets (2.15).

Last, we should say that the term on the last line of Eq. (3.12) reads

$$\sigma = -\sum_{n=2}^{\infty} \left[-2n\Big(\eta(Mr_+) - \eta(Mr_-)\Big) - \frac{F_2(1)}{n^2} \right], \tag{A24}$$

where

$$-2n\Big(\eta(Mr_+) - \eta(Mr_-)\Big) \sim \ln(r_+/r_-)^{-2n}, \tag{A25}$$

while, with the notation of Eqs. (A6)–(A11), one has

$$F_2(1) = 2D_2(1) + 2M_2(1,1) = -1. \tag{A26}$$

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